Deep Generative Models

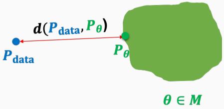
Lecture 4: Maximum Likelihood Learning

Aditya Grover

UCLA

Learning a generative model

- **Given:** a training set of examples, e.g., images of dogs



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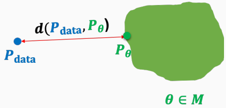

- **Goal:** learn a probability distribution $p(x)$ over images x
  - **Generation:** If we sample $x_{new} \sim p(x)$, x_{new} should look like a dog (*sampling*)
  - **Density estimation:** $p(x)$ should be high if x looks like a dog, and low otherwise (*anomaly detection*)
  - **Unsupervised representation learning:** We should be able to learn what these images have in common, e.g., ears, tail, etc. (*features*)
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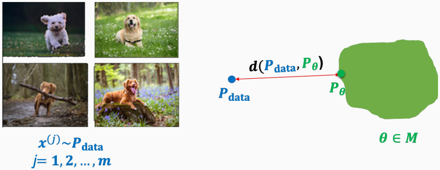

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- First question: how to represent $p_\theta(x)$. Second question: **how to learn it.**

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3. Structure or knowledge discovery: we are interested in the model itself
    - How do some genes interact with each other?
    - What causes cancer?

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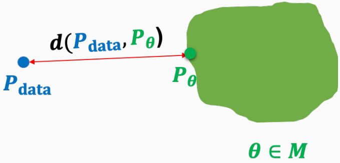

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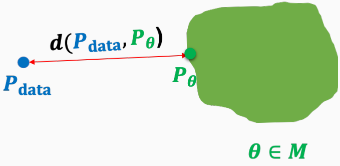

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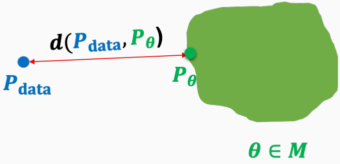

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- Notice that KL-divergence is **asymmetric**, i.e., $D(p\|q) \neq D(q\|p)$
- Measures the expected number of extra bits required to describe *samples from* $p(\mathbf{x})$ using a code based on q instead of p

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  - Batch multiple samples together
  - Use a short sequence of bits to encode $HHHH$ (common) and a long sequence for $TTTT$ (rare).
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  - Like Morse code: $E = \bullet$, $A = \bullet -$, $Q = - - \bullet -$
- KL-divergence: if your data comes from p, but you use a scheme optimized for q, the divergence $D_{KL}(p||q)$ is the number of *extra* bits you'll need on average

Learning as density estimation

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  - Because of log, samples $\mathbf{x}$ where $P_\theta(\mathbf{x}) \approx 0$ weigh heavily in objective

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- Problem: In general we do not know P_{data}.

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- Approximate the expected log-likelihood

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Main idea in Monte Carlo Estimation

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Thus, variance of the estimator can be reduced by increasing the number of samples.

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Single variable example: A biased coin

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- Example learning task: How should we choose $P_\theta(x)$ from $\mathcal{M}$ if 60 out of 100 tosses are heads in $\mathcal{D}$?

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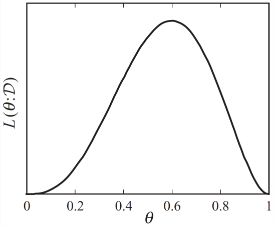

- Optimize for θ which makes $\mathcal{D}$ most likely. What is the solution in this case?

MLE scoring for the coin example: Analytical derivation

Distribution: $P_\theta(x = H) = \theta$ and $P_\theta(x = T) = 1 - \theta$

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Given an autoregressive model with n variables and factorization

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- We no longer have a closed form solution

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1. Initialize $\theta^0 = (\theta_1, \cdots, \theta_n)$ at random
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Non-convex optimization problem, but often works well in practice

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- Thus, we typically restrict the **hypothesis space** of distributions that we search over

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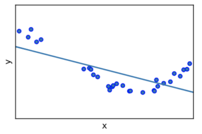

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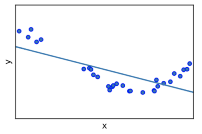

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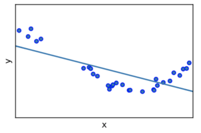

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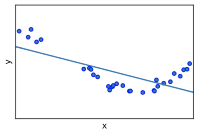

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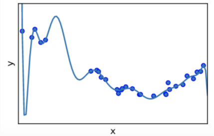

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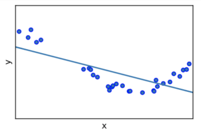

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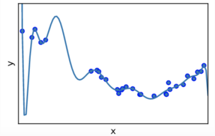

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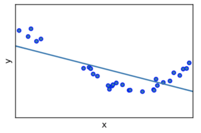

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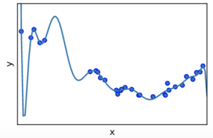

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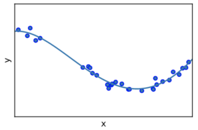

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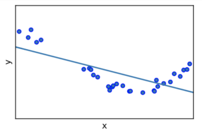

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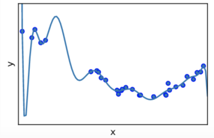

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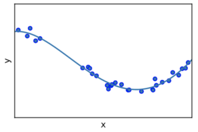

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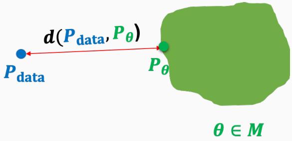

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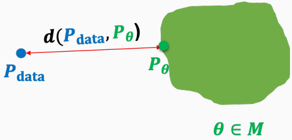

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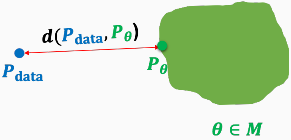

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- Other ways of measuring similarity are possible (Generative Adversarial Networks, GANs)