# Deep Generative Models

Lecture 7: Normalizing Flows

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UCLA

# Recap of likelihood-based learning so far:

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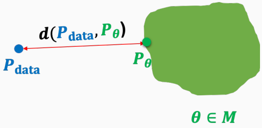

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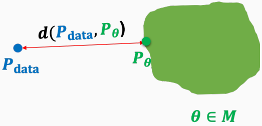


- Model families:
  - Autoregressive Models: $p_\theta(\mathbf{x}) = \prod_{i=1}^{n} p_\theta(x_i | \mathbf{x}_{<i})$
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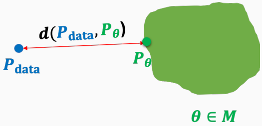


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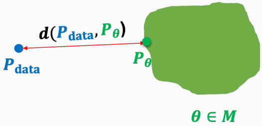


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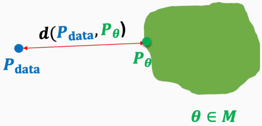


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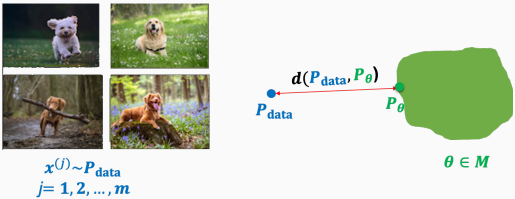


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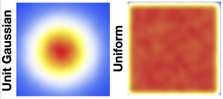

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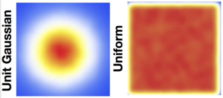


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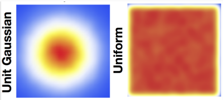


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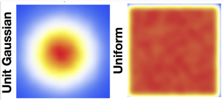


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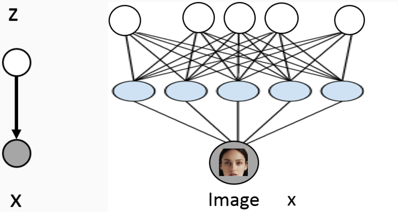
- **Key idea behind flow models**: Map simple distributions (easy to sample and evaluate densities) to complex distributions through an **invertible transformation**.

# Variational Autoencoder

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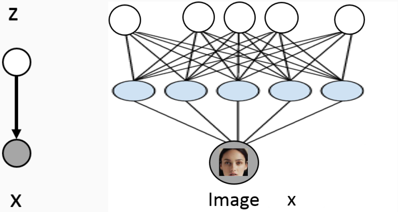


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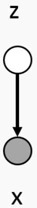


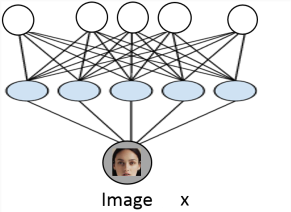


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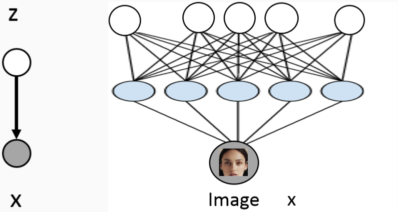


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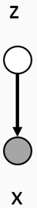


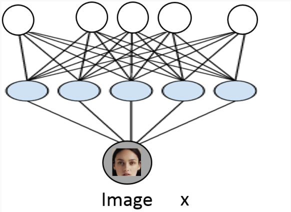


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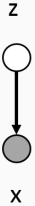


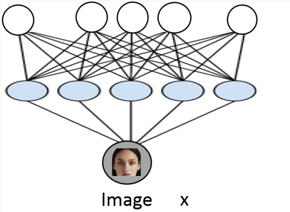


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4. What if we could easily "invert" $p(\mathbf{x} \mid \mathbf{z})$ and compute $p(\mathbf{z} \mid \mathbf{x})$ by design? How? Make $\mathbf{x} = f_\theta(\mathbf{z})$ a deterministic and invertible function of $\mathbf{z}$

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- To get correct result, need to use **change of variables formula**

# Change of Variables formula

- **Change of variables (1D case)**: If $X = f(Z)$ and $f(\cdot)$ is monotone with inverse $Z = f^{-1}(X) = h(X)$, then:

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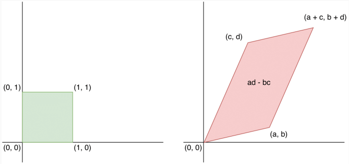


**Figure 1:** The matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ maps a unit square to a parallelogram

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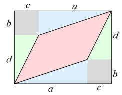
$$\det(A) = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc$$

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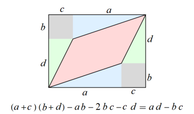

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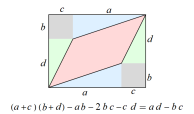


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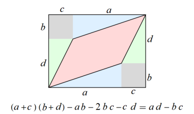


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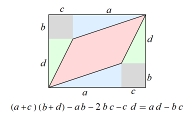


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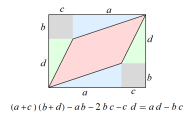


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&= p_{Z_1,Z_2}(z_1, z_2) \left| \det \begin{pmatrix} \frac{\partial u_1(z_1,z_2)}{\partial z_1} & \frac{\partial u_1(z_1,z_2)}{\partial z_2} \\ \frac{\partial u_2(z_1,z_2)}{\partial z_1} & \frac{\partial u_2(z_1,z_2)}{\partial z_2} \end{pmatrix} \right|^{-1} \text{(forward)}
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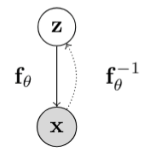

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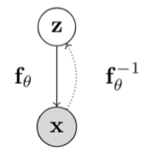


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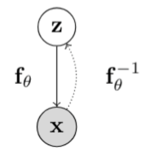


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- Note: $\mathbf{x}, \mathbf{z}$ need to be continuous and have the same dimension.

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(Note: determininant of product equals product of determinants)

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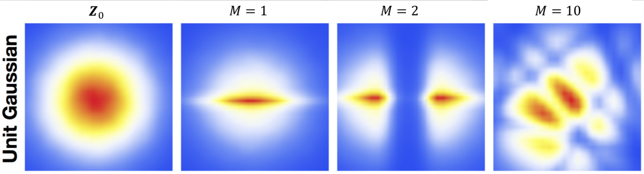

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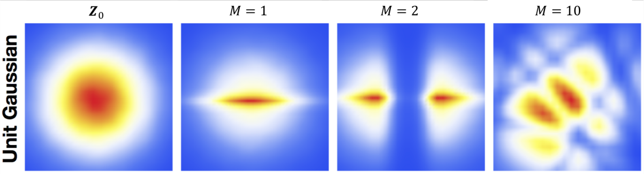


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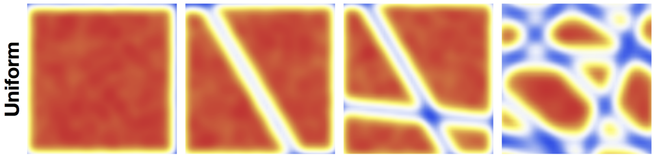

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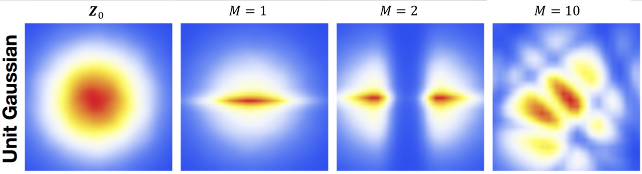


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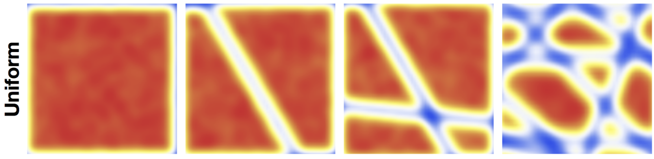


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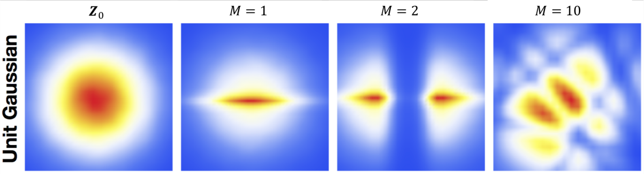

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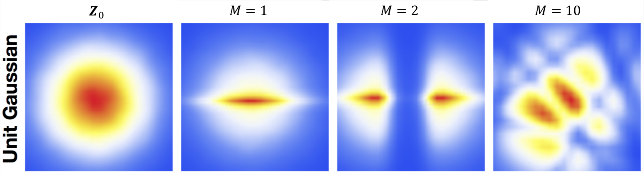


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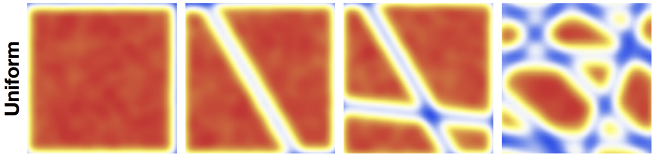

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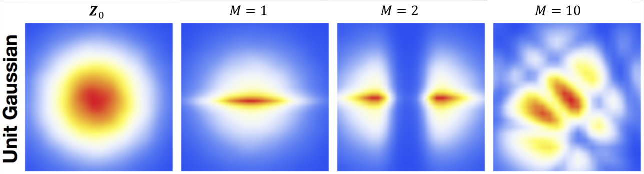


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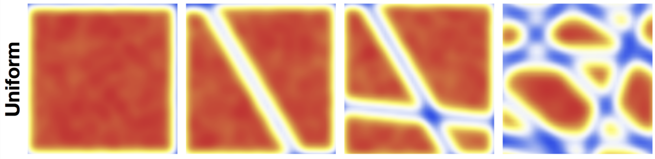


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**Next lecture:** More invertible transformations for high dimensions!